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The Fundamental Theorem of Algebra and the Interpolating Envelope for Totally Positive Perfect Splines

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INTRODUCTION

This paper deals with a generalization of polynomial perfect splines. Perfect splines play an important role in many optimization problems which arise when one is dealing with the uniform norm. For example, in the problems of n-widths [12, 17] and optimal interpolation [2, 5, 7, 15] in $L^{\infty}[0, 1]$, perfect splines are used to describe the solutions. In another area, Micchelli [11] has recently developed the relationship between L_1 approximation and perfect splines.

We use the following definitions.

(1) A set of m functions $\{u_1,..., u_m\} \subset C[0, 1]$ is said to be a weak Tchebycheff system of order m if they are independent and if for each set $0 < t_1 < \cdots < t_m < 1$, $\det\{u_i(t_j);\ i, j = 1,..., m\} \geqslant 0$.

(2) A set of m functions $\{u_1,..., u_m\} \subset C^{m-1}[0, 1]$ is said to form an Extended Complete Tchebycheff system if for each $1 \leqslant k \leqslant m$ and each $1 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant 1$, $\det\{u_i(t_j): i, j = 1,..., k\} > 0$. This is more precisely defined in [8, p. 5].

Let $\{k_1,..., k_r\} \subset C[0, 1]$ and let $K(x, y)$ be a continuous function from $[0, 1] \times [0, 1]$ to the reals. For each $1 \leqslant i_1 < \cdots < i_s \leqslant r$; $0 < y_1 < \cdots < y_l < 1$, and $0 < t_1 < \cdots < t_{s+l} < 1$ define

$$K\begin{pmatrix} i_1,..., i_s, y_1,..., y_l \\ t_1,..., t_s,..., t_{s+l} \end{pmatrix} = \det \begin{bmatrix} k_{i_1}(t_1) \cdots & k_{i_1}(t_{s+l}) \\ \vdots & \vdots \\ k_{i_s}(t_1) \cdots & k_{i_s}(t_{s+l}) \\ K(t_1, y_1) \cdots & K(t_{s+l}, y_1) \\ \vdots & \vdots \\ K(t_1, y_l) \cdots & K(t_{s+l}, y_l) \end{bmatrix}.$$

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We follow Micchelli and Pinkus [12] in making the following *Basic Assumptions* on the functions defined above.

(1) For each set of points, $0 < y_1 < \cdots < y_m < 1$, the functions $\{k_1(t),..., k_r(t), K(t, y_1),..., K(t, y_m)\}$ are independent over $[0, 1]$.

(2) $k_1(x),..., k_r(x)$ form a Tchebyscheff system over $(0, 1)$.

(3) Each determinant,

$$K\begin{pmatrix} i_1,..., i_s, y_1,..., y_l \\ t_1,..., t_s,..., t_{s+l} \end{pmatrix},$$

as defined above is non-negative.

(4) For arbitrary $0 < t_1 < \cdots < t_s < 1$; $K(t_1, y),..., K(t_s, y)$ are linearly independent.

For the remainder of this paper we assume all functions $k(x), K(x, y)$ satisfy these *Basic Assumptions*, unless we specify to the contrary.

If $k_1 > 0$ we define the differential operator $(Df)(x) \equiv (d/dx)(f(x)/k_1(x))$.

DEFINITION. A totally positive perfect spline is any function of the form

$$P(t) = \sum_{i=1}^{r} a_i k_i(t) + c \sum_{j=1}^{s} (-1)^j \int_{\xi_j}^{\xi_{j+1}} K(t, y)\, dy, \tag{1}$$

where $0 = \xi_0 < \xi_1 < \cdots < \xi_{s+1} = 1$, and where $k(x)$, $K(x, y)$ satisfy the *Basic Assumptions*.

We are able to demonstrate in this paper that such functions are unique with respect to interpolation. Using this uniqueness many classical results can be extended to this broad class of functions, and some new results involving polynomial perfect splines can be established. For example, let n be a fixed integer greater than r, then for a given pair of numbers $(\hat{t}, \alpha)$ and a given set of n distinct numbers $\{t_i\}_{i=1}^n \subset (0, 1)$ there is a unique $P(t)$ of the above form where $s \leqslant n - r$ such that

$$P(\hat{t}) = \alpha, \qquad P(t_i) = 0, \qquad i = 1,...,)n.$$

Here $\alpha \neq 0$ and $\hat{t} \in (0, 1)$ is distinct from $\{t_i\}_{i=1}^n$. We also develop some extensions of this theorem for multiple zeros.

Another result that we obtain has application in n-widths. It concerns the uniqueness of a perfect splines of the form (1) with $c = 1$ which alternates a maximal number of times, [5, 12, 17]. Further for L_1 approximation by weak Tchebycheff systems the results of Micchelli [11] are extended. Finally the interpolating envelope development of Micchelli and Miranker [13] is generalized.

UNIQUENESS OF INTERPOLATION

We begin this section by stating a result that can be obtained by modifying the proof of the corresponding result in Micchelli and Pinkus [12].

LEMMA 1. *For any set of constants* $\{\alpha_1,..., \alpha_n\}$, *any set of knots* $0=\xi_0<\xi_1<\cdots<\xi_{n-r+1}=1$, *and any* $h\in L^\infty$ *such that* $y\in(\xi_j, \xi_{j+1})\Rightarrow (-1)^j h(y)\geqslant 0$ $(j=0, 1,..., n-r)$, *the function*

$$g(x)=\sum_{j=1}^{r}\alpha_j k_j(x)+\int_0^1 h(y)\,K(x, y)\,dy+\sum_{j=r+1}^{n}\alpha_j K(x, \xi_{j-r}) \tag{2}$$

has at most n distinct zeros in (0, 1) *in each of the following two cases*:

(A)(a) *For arbitrary* $0<\xi_1<\cdots<\xi_m<1$; $k_1(x),..., k_r(x)$, $K(x, \xi_1),..., K(x, \xi_m)$ *is a Tchebycheff system.*

(b) $h(y)$ *is non-zero on a set of positive measure.*

(B)(a) $k(x)$, $K(x, y)$ *satisfy* Basic Assumptions (1), (3) *and* (4).

(b) $h(y)$ *is non-zero a.e.*

With respect to the existence of perfect splines which interpolate specified data, Karlin in his basic paper [7] on this subject has proven by the use of non-linear mapping theorems, that one can always interpolate with totally positive perfect splines. de Boor [2] has given a short proof of Karlin's result using linear functionals. We state this result as Theorem 1.

THEOREM 1. *For a given set of points* $0<x_1<\cdots<x_{n+1}<1$ *and a set of data* $\{\beta_i\}_{i=1}^{n+1}$, *there exist* $\alpha_1,..., \alpha_r$, c, ξ_j *and an integer* $q\geqslant 0$ *such that*

$$\sum_{j=1}^{r}\alpha_j k_j(x_i)+c\left[\sum_{j=0}^{n-r-q}(-1)^j\int_{\xi_j}^{\xi_{j+1}}K(x_i, y)\,dy\right]=\beta_i \qquad (i=1,..., n+1),$$

where $0=\xi_0<\xi_1<\cdots<\xi_{n-r-q}<\xi_{n-r-q+1}=1$, *when* $k(x)$, $K(x, y)$ *satisfy the* Basic Assumptions.

LEMMA 2. *Let* $k(x)$, $K(x, y)$ *satisfy the* Basic Assumptions, and furthermore let $k_1>0$ and $DK(x, y)$ *satisfy* Basic Assumption 4, *then any perfect spline of the form*:

$$P(t)=\sum_{i=1}^{r}a_i k_i(t)+c\sum_{i=0}^{n-r}\int_{\xi_i}^{\xi_{i+1}}K(t, y)\,dy+\sum_{i=1}^{n-r}b_i K(t, \xi_i), \tag{3}$$

where $0=\xi_0<\xi_1<\cdots<\xi_{n-r+1}=1$, *has at most n zeros, counting* *multiplicities up to order* 2 *in* (0, 1) *and* $D(P)$ *has at most* $n-1$ *distinct zeros in* (0, 1).

Proof. First consider any set: $\{0 < y_1 < \cdots < y_s < 1\}$. We claim that the functions $\{Dk_2,..., Dk_r, DK(\cdot, y_1),..., DK(\cdot, y_s)\}$ are independent. For if not there is a set of numbers $\{\alpha_2,..., \alpha_r, \beta_1,..., \beta_s\}$, all not zero, and a constant of integration c such that

$$\sum_{i=2}^{r} \alpha_i k_i + \sum_{i=1}^{s} \beta_i K(\cdot, y_i) \equiv ck_1.$$

This contradicts our basic independence hypothesis and thus the functions generated by the differential operator D are independent. For each $\varepsilon > 0$, let $k_i^{(\epsilon)}$ and $K^{(\epsilon)}(\cdot, y_j)$ be the Gaussian Transform of k_i and $K(\cdot, y_j)$, respectively. It is well known (see Theorem 2) that the functions $\{k_1^{(\epsilon)},..., k_r^{\epsilon}, K^{(\epsilon)}(\cdot, y_1),..., K^{(\epsilon)}(\cdot, y_s)\}$ form an Extended Complete Tchebycheff System. Thus by the reduction result of Karlin, Studden [8, see Eq. (1.4), p. 377], with $D_\epsilon F(x) = (d/dx)(F(x)/k_1^{(\epsilon)}(x))$

$$D_\epsilon k_2^{(\epsilon)},..., D_\epsilon k_r^{(\epsilon)},..., D_\epsilon K^{(\epsilon)}(\cdot, y_s))$$

form an Extended Complete Tchebycheff System. Letting $\varepsilon \to 0$ in the above system and applying the fact that the limit functions are independent shows that the limit functions

$$Dk_2,..., Dk_r, DK(\cdot, y_1),..., DK(\cdot, y_s)$$

form a weak Tchebycheff system. Thus Lemma 1 is applicable to $D(P)$ and hence $D(P)$ has at most $n-1$ distinct zeros in (0, 1). Note that if P had $n+1$ zeros in (0, 1) including multiplicities up to order 2, it would follow by Rolle's Theorem that $D(P)$ has at least n distinct zeros in (0, 1), which is a contradiction. This completes the proof.

Remark 1. If $k_1 > 0$ and $Dk_2 > 0$ and

$$\frac{\partial}{\partial x} \frac{1}{Dk_2(x)} \frac{\partial}{\partial x} \frac{K(x, y)}{k_1(x)}$$

satisfies *Basic Assumption* (4) it follows in a similar fashion that P has at most n zeros in (0, 1) counting multiplicities up to order three, and $D(P)$ has at most $n-1$ zeros in (0, 1) counting multiplicities up to order two.

LEMMA 3. *If* $P(t)$ *of the form* (1) *has* n *zeros*, $0 < t_1 < \cdots < t_n < 1$, *then*

$$K\begin{pmatrix} 1,..., r, \xi_1,..., \xi_{n-r} \\ t_1,..., t_r, t_{r+1},..., t_n \end{pmatrix} > 0.$$

Proof. See Micchelli and Pinkus [12].

Remark 2. If, for example, $k_1 > 0$ (and $Dk_2 > 0$), then the above inequality is valid in the case where at most two (three) t_i are permitted to be the same. The proof of these results employs Lemma 2 and proceeds analogously to the derivation of Lemma 3.

THEOREM 2 (Uniqueness of Interpolation). *Consider any perfect spline of the form*

$$P(t) = \sum_{i=1}^{r} a_i^* k_i(t) + c^* \sum_{i=0}^{n-r} (-1)^i \int_{\xi_i^*}^{\xi_{i+1}^*} K(t, y)\, dy, \tag{4}$$

where $0 < \xi_0^* < \xi_1^* < \cdots < \xi_{n-r+1}^* = 1$; $c^* \neq 0$. *If, for a given set* $0 < t_1 < \cdots < t_{n+1} < 1$, *there is a subset of* n *of these* t*'s, say,* $0 < t_{i_1} < \cdots < t_{i_n} < 1$, *such that*

$$K\begin{pmatrix} 1,\ldots, r, \xi_1^*,\ldots, \xi_{n-r}^* \\ t_{i_1}, t_{i_2},\ldots, t_{i_n} \end{pmatrix} > 0 \tag{5}$$

then the following proposition is valid: For any perfect spline of the form

$$Q(t) = \sum_{i=0}^{r} \hat{a}_i k_i(t) + \hat{c} \sum_{j=0}^{s} (-1) \int_{\hat{\xi}_j}^{\hat{\xi}_{j+1}} K(t, y)\, dy, \tag{6}$$

where $0 = \xi_0 < \xi_1 < \cdots < \xi_{s+1} = 1$ *and where* $s \leqslant n - r$, *which agrees with* P *on the set* $\{t_i\}_{i=1}^{n+1}$, *we have*

$$P \equiv Q.$$

Proof. We first calculate the Jacobian of $P(t)$ with respect to the parameters $\{a_i^*, c^*, \xi_j^*\}$. Let

$$g_i(t) \equiv \frac{\partial P(t)}{\partial a_i^*} = k_i(t) \qquad (i = 1,\ldots, r),$$

$$g_{r+j}(t) \equiv \frac{\partial P(t)}{\partial \xi_j^*} = 2(-1)^{j+1} c^* K(t, \xi_j^*) \qquad (j = 1,\ldots, n-r),$$

$$g_{n+1}(t) \equiv \frac{\partial p(t)}{\partial c^*} = \sum_{j=0}^{n-r} (-1)^j \int_{\xi_j^*}^{\xi_{j+1}^*} K(t, y)\, dy.$$

We claim that the Jacobian J of $P(t)$ at the points $\{t_i\}_{i=1}^{n+1}$ is non-zero; that is,

$$J \equiv |\, g_i(t_j); (i, j = 1,\ldots, n+1)| \neq 0.$$

If this claim is not valid there is a non-zero vector $(b_1,..., b_{n+1}) \subset R^{n+1}$ so that the function

$$g(t) \equiv \sum_{i=1}^{n+1} b_i g_i(t)$$

has zeros at the $\{t_i\}_{i=1}^{n+1}$. If $b_{n+1} \neq 0$, Lemma 1 states that $g(t)$ can have at most n zeros. Thus $b_{n+1} = 0$. But in this case our hypothesis (5) implies that $g(t)$ cannot vanish at $\{t_j\}_{j=1}^{n}$, which is a contradiction. Hence $J \neq 0$.

We now consider the Gaussian Transform of P and Q for $|\varepsilon| > 0$. For any f, the Gaussian transform $f^{(\varepsilon)}$ of f is formally,

$$f(t; \varepsilon) = f^{(\varepsilon)}(t) = \frac{1}{\sqrt{2\pi}\,|\varepsilon|} \int_{-\infty}^{\infty} \exp\left(\frac{-(z-t)^2}{2\varepsilon^2}\right) f(z)\, dz$$

with $f^{(0)}(t) = f(t)$. For any set $0 < y_1 < \cdots < y_s < 1$, it is well known [8, p. 15] that for each $|\varepsilon| > 0$ the set of $s + r$ functions $\{k_1^{(\varepsilon)},..., k_r^{(\varepsilon)}, K^{(\varepsilon)}(\cdot, y_1),..., K^{(\varepsilon)}(\cdot, y_s)\}$ form an Extended Complete Tchebycheff System. In addition as $\varepsilon \to 0$, the functions and their partial derivatives converge uniformly to the corresponding k_i or $K(\cdot, y_j)$ on any compact subset of (0, 1) up to the degree of smoothness that the limit function possesses. From the linear properties of the Gaussian Transform it follows that:

$$P^{(\varepsilon)}(t) = \sum_{i=0}^{r} a_i^* k_i^{(\varepsilon)}(t) + c^* \sum_{i=0}^{n-r} (-1)^i \int_{\xi_i^*}^{\xi_{i+1}^*} K^{(\varepsilon)}(t, y)\, dy. \qquad (7)$$

Assume there is a Q of the form (6), different from P, such that $P(t_i) = Q(t_i)$ $(i = 1,..., n+1)$. In order to arrive at a contradiction consider the system of $n + 1$ non-linear equations in $n + 1$ unknowns (which are the components of A):

$$F_j(A; \varepsilon) \equiv \sum_{i=0}^{r} a_i k_i^{(\varepsilon)}(t_j) + c \sum_{j=0}^{n-r} (-1)^j \int_{\xi_j}^{\xi_{j+1}} K^{(\varepsilon)}(t_j, y)\, dy = \alpha_j,$$

$$j = 1,..., n+1, \quad (8)$$

where $A = (a_1,..., a_r,\ \xi_1,..., \xi_{n-r}, c)$. Note that when $\varepsilon = 0$ and $\alpha_j = P(t_j)$ $(j = 1,..., n+1)$, we have a solution $A^* = (a_1^*,..., a_r^*, \xi_1^*,..., \xi_{n-r}^*, c^*)$ to (8). Now consider $\alpha = (\alpha_1,..., \alpha_{n+1})$ and ε as parameters in this non-linear system. Since the Jacobian of the system $J \neq 0$ at $\varepsilon = 0$, $A = A^*$ and $\alpha = \alpha^* \equiv (P(t_1),..., P(t_{n+1}))$ we can invoke the *implicit function theorem* to show that there is a neighborhood V of $(\alpha^*, 0) \subset R^{n+2}$ such that

(I) For each (α, ε) in V there exists a solution $A(\alpha, \varepsilon)$ of (8) with the last component $c(\alpha, \varepsilon) \neq 0$.

(II) There is a number $d > 0$ with the property that for each $A(\alpha, \varepsilon) \in V$

$$\sup_{t \in [0,1]} |P^{(\epsilon)}(A(\alpha, \varepsilon), t) - Q(t)| \geqslant d.$$

Here $P^{(\epsilon)}(A(\alpha, \varepsilon), t)$ is the perfect spline of the form (7) with the parameter set $A(\alpha, \varepsilon)$.

For $|\varepsilon| > 0$ consider the Gaussian Transform $Q^{(\epsilon)}(t)$ of $Q(t)$. By the continuity properties of the Gaussian Transform, there is an $\varepsilon_0 > 0$ such that

(a) $|\varepsilon'| < \varepsilon_0 \Rightarrow (Q^{(\epsilon')}(t_1),..., Q^{(\epsilon')}(t_{n+1}), \varepsilon') \in V$.

(b) There is a number $\rho > 0$ such that $|\varepsilon'| < \varepsilon_0$ and

$$(\alpha, \varepsilon) \in V \Rightarrow \max_{t \in [0,1]} |Q^{(\epsilon)}(t) - P^{(\epsilon)}(A(\alpha, \varepsilon), t)| \geqslant \rho.$$

Hence for $0 < |\varepsilon| < \varepsilon_0$,

$$Q^{(\epsilon)}(t_j) = P^{(\epsilon)} A(Q^{(\epsilon)}(t_1),..., Q^{(\epsilon)}(t_{n+1})), \varepsilon), t_j) \qquad (j = 1,..., n+1)$$

and

$$Q^{(\epsilon)} \not\equiv P^{(\epsilon)}(A(Q^{(\epsilon)}(t_1),..., Q^{(\epsilon)}(t_{n+1})), \varepsilon), \cdot).$$

Call the function on the right above $P^{(\epsilon)}$. It is easy to see that $P^{(\epsilon)} - Q^{(\epsilon)}$ has the form

$$(P^{(\epsilon)} - Q^{(\epsilon)})(t) = \sum_{i=1}^{r} a_i k_i^{(\epsilon)}(t) + \int_0^1 h(y) K^{(\epsilon)}(t, y)\, dy,$$

where $h(y)$ has at most $n - r$ sign changes. According to Lemma 1, $P^{(\epsilon)} - Q^{(\epsilon)}$ can have at most n zeros which is a contradiction. Thus $P \equiv Q$. ∎

COROLLARY 1 (Fundamental Theorem of Algebra for Totally Positive Perfect Splines). *For a set of n distinct points* $0 < t_1 < \cdots < t_n < 1$ *and a pair of real numbers* (t, α), *where* $t \in (0, 1) - \{t_i\}_{i=1}^n$ *and* $\alpha \neq 0$, *there exists a unique P of the form* (4) *such that*

$$P(t_i) = 0, \qquad i = 1,..., n,$$

$$P(t) = \alpha.$$

Proof. The existence of such an interpolating $P(t)$ of the form (4) with knots $\{\xi_i^*\}$ follows from Theorem 1 where $c^* \neq 0$. (Since $k_1,..., k_r$ form a

Tchebycheff system, it follows that for all such solutions, the coefficient $c^* \neq 0$.) By Lemma 3.

$$K\begin{pmatrix} 1,..., r, \xi_1^*, \xi_2^*,..., \xi_{n-r}^* \\ t_1,..., t_r,..., t_n \end{pmatrix} > 0$$

Hence the *Uniqueness of Interpolation Theorem* applies and $P(t)$ is the unique interpolating perfect spline of the form (6). ∎

Remark 3. If $k_1 > 0$, one can with the help of Remark 1, extend the *Fundamental Theorem of Algebra* to the case where we allow zeros of multiplicity two. Indeed with appropriate hypothesis (which are valid in the classical problem of polynomial perfect splines) these results can be extended to even higher multiplicities. For example, if one assumes in addition that $Dk_2 > 0$ we can allow zeros up to multiplicity three.

As another application of this theorem let us recover Karlin's elegant result [7] on interpolating oscillatory data with polynomial perfect splines. To be specific let $r = m \geqslant 2$, $k_i(t) = t^{i-1}$, $i = 1,..., m$; $K(t, y) = (t - y)_+^{m-1}$, and let $\{\alpha_i\}_{i=1}^{n+1}$ be a set of given data associated with a set of $n + 1$ points $\{t_i\}_{i=1}^{n+1}$, where $0 \leqslant t_1 < t_2 < \cdots < t_{n+1} \leqslant 1$. Further let $[\alpha_l,..., \alpha_s]$ be the $(s - l)$th ordered divided difference of the data $\{\alpha_i\}_{i=l}^{s}$ with respect to the points $\{t_i\}_{i=1}^{s}$.

COROLLARY 2 (Karlin, [7]). *If $m \geqslant n - m$ and for some s, where $n \geqslant s \geqslant 0$, $[\alpha_l, \alpha_{l+1},..., \alpha_{l+s}][\alpha_{l+1},..., \alpha_{l+s+1}] < 0$ $(i = 1,..., n - s)$ then there exists a unique perfect spline P with at most $n - m$ knots which interpolates the given data, that is $P(t_i) = \alpha_i$ $i = 1,..., n + 1$.*

Proof. According to Theorem 5.1 of [Karlin, 7] there is a perfect spline with exactly $n - m$ distinct knots which interpolates the given data. Further if $0 < \xi_1 < \cdots < \xi_{n-m} < 1$ are the knots of P,

(a) $t_\nu < \xi_\nu < t_{\nu+m+1}$, $\qquad \nu = 1,..., n - m.$

Now let

(b) $x_\nu = t_\nu$, $\qquad \nu = 1,..., n - m.$

(c) $x_\nu = t_{\nu+1}$, $\qquad \nu = n - m + 1,..., m.$

From (a) and (b),

(*d*) $x_\nu = t_\nu < \xi_\nu$, $\qquad \nu = 1,..., n - m.$

Further from (a) and (c) and the fact that $m > n - m$

(e) $\xi_\nu < t_{\nu+m+1} = x_{\nu+m}$, $\qquad \nu = 1,..., n - m.$

Combining (d) and (e) we have

(f) $x_\nu < \xi_\nu < x_{\nu+m}$, $\qquad \nu = 1,\dots, n-m$.

But as is well known [6, p. 503], the above inequalities imply

$$K\begin{pmatrix} 1,\dots, m, \xi_1,\dots, \xi_{n-m} \\ x_1,\dots, x_n \end{pmatrix} > 0.$$

Thus Theorem 2 is applicable and P is the unique interpolant. ∎

L_1 APPROXIMATION

In this section we consider the problem of approximating a function $f \in C[0, 1]$ in the L_1 norm by elements of an m-dimensional weak Tchebycheff subspace U of $C[0, 1]$ with a basis $u_1,\dots, u_m$. Specifically we characterize the element $u \in U$ which minimizes

$$\|f-u\| \equiv \int_0^1 |f(x) - u(x)|\, dx.$$

Following Micchelli we define the convex cone of U,

$$K(U) = \left\{ f \in C[0, 1]: 0 \leqslant y_1 < \cdots < y_{m+1} \leqslant); U\begin{pmatrix} 1,\dots, m, f \\ y_1,\dots, y_m, y_{m+1} \end{pmatrix} \geqslant 0 \right\}$$

and for each set $0 < y_1 < \cdots < y_m < 1$, let

$$U[y_1,\dots, y_m] = \{(f(y_1),\dots, f(y_m)): f \in K(U)\}.$$

Here

$$U\begin{pmatrix} 1,\dots, m \\ y_1,\dots, y_m \end{pmatrix} = \det\{u_i(y_j)\}.$$

THEOREM 3. *For each sequence* $0 < y_1 < \cdots < y_m < 1$, *assume that* $U[y_1,\dots, y_m]$ *contains a basis for* R^m, *then there is a unique set of nodes* $0 = \xi_0 < \xi_1 < \cdots < \xi_s < \xi_{s+1} = 1$, *where* $s \leqslant m$ *such that*

$$\sum_{j=0}^{s} (-1)^j \int_{\xi_j}^{\xi_{j+1}} u_i(y)\, dy = 0 \qquad (i = 1,\dots, m). \tag{9}$$

Further $s = m$.

Proof. We can apply the Hobby–Rice Theorem [4] or Theorem 1, to demonstrate that a set of knots $\{\xi_j\}_{j=1}^{s}$ exist which satisfy (9). Micchelli [11] has proven that $s = m$ and that

$$U\begin{pmatrix} 1,..., m \\ \xi_1,..., \xi_m \end{pmatrix} \neq 0.$$

A simple modification of Theorem 2 with the correspondence,

$$u_i(y) = K(t_i, y)$$

yields the fact that the $\{\xi_i\}_{i=1}^{m}$ are unique. ∎

Combining the above theorem and Micchelli [11, Theorem 1] we have:

THEOREM 4. *Assume that for each sequence* $0 < y_1 < \cdots < y_m < 1$, $U[y_1,..., y_m]$ *contains a basis for* R^m. *Then there is a unique set of points* $0 < \xi_1 < \cdots < \xi_m < 1$ *such that for every* $f \in K(U)$, *the element* $u \in U$ *which is closest to* f *in the* L_1 *norm is completely determined by the conditions*

$$u(\xi_i) = f(\xi_i) \qquad (i = 1,..., m).$$

Micchelli [11], has obtained the above result using a stronger set of hypotheses.

OSCILLATORY PERFECT SPLINES

In this section we establish the uniqueness of the totally positive perfect spline which alternates maximally over [0, 1]. Our theorem extends the known classical result for polynomial perfect splines [5]. One is referred to the papers of Karlin, Micchelli, Pinkus, and Tihomirov [5, 7, 12, 17], for the relationship of this problem to certain extremal problems in $L^\infty[0, 1]$.

THEOREM 5. *Assume* $r \geqslant 2$, $k_1 = 1$ *and* $(d/dx)\, k_2 > 0$, *and*

$$\frac{\partial}{\partial x}\frac{1}{k_2'(x)}\frac{\partial}{\partial x}K(x, y)$$

satisfies Basic Assumption 4, *then there is exactly one perfect spline of the form*

$$P(t) = \sum_{i=1}^{r} a_i k_i(t) + \sum_{j=0}^{n-r} (-1)^j \int_{\xi_j}^{\xi_{j+1}} K(t, y)\, dy,$$

where $0=\varepsilon_0<\xi_1<\cdots<\xi_{n-r+1}=1$ *so that for some* $E>0$,

(I) $-E \leqslant P(t) \leqslant E$, $t \in [0, 1]$.

(II) *There exist* $n+1$ *points* $0<x_1<\cdots<x_{n+1}<1$ *such that equality occurs in* (I) *only at the* $\{x_i\}$. *Further* $P(x_i)=(-1)^{i-r+1}E$, $i=1,\ldots,n+1$.

Proof. The existence of such an alternating perfect spline has been established in [12]. Let P^* be such an alternating spline with parameters $A^*=(a_1^*,\ldots,a_r^*,\xi_1^*,\ldots,\xi_{n-r}^*)$, $E=E^*$, and alternating points $0\leqslant x_1^*<\cdots<x_{n+1}^*\leqslant 1$. Note that by Lemma 2, $x_1^*=0$ and $x_{n+1}^*=1$. Consider the system of $2n$ equations in $2n$ unknowns

$$\begin{aligned} P(A, x_i) &\equiv \sum_{j=1}^{r} a_j k_j(x_i) + \sum_{j=1}^{n-r} (-1)^j \int_{\xi_j}^{\xi_{j+1}} K(x_i, y)\, dy \\ &= (-1)^{i-r+1} E \qquad (i=1,\ldots,n+1), \end{aligned} \tag{10}$$

$$\frac{d}{dx} P(A, x) \Big|_{x=x_i} = 0 \qquad (i=2,\ldots,n). \tag{11}$$

Here the $2n$ unknowns are $A=(a_1,\ldots,a_r,\xi_1,\ldots,\xi_{n-r})\subset R^n$, $0<x_2<\cdots<x_n<1$ and $E\subset R$. Further $\xi_0\equiv 0\equiv x_1$ and $\xi_{n-r+1}\equiv 1\equiv x_{n+1}$. Note that P^* yields a solution of (10) and (11) and by Remark 1, $d^2/dx^2 P^*(x_i^*)\neq 0$ $(i=2,\ldots,n)$.

Thus by the *implicit function theorem* for each i in (11) we can solve for x_i as a function of A in a neighborhood of A^*. We restrict ourselves from now on to this neighborhood and deal exclusively with system (10) where each x_i is a function of A. Taking into account the fact that $d/dx\, P(A, x_i(A))=0$, the Jacobian of (10) with respect to the parameters A and E is

$$J = \begin{vmatrix} 1,\ldots, k_r(x_1), & 2(-1)K(x_1,\xi_1),\ldots, 2(-1)^{n-r}K(x_1,\xi_{n-r}), & (-1)^{1-r} \\ \cdot & \cdot \qquad\qquad \cdot & (-1)^{2-r} \\ \cdot & \cdot \qquad\qquad \cdot & \cdot \\ \cdot & \cdot \qquad\qquad \cdot & \cdot \\ 1,\ldots, k_r(x_{n+1}), & 2(-1)K(x_{n+1},\xi_1),\ldots, 2(-1)^{n-r}K(x_{n+1},\xi_{n-r}), & (-1)^{n+1-r} \end{vmatrix}.$$

We want to show $J\neq 0$. Hence we can factor out powers of 2 and -1 from the columns of J and subtract the ith row from the $i+1$ row $(i=1,\ldots,n)$; to obtain a determinant $\hat{J}$ on which we can equivalently test whether the Jacobian is non-zero. Expanding $\hat{J}$ by the first column, applying the fundamental theorem of calculus n times, and using the multilinear properties of a determinant we find that

$$\hat{J} = \int_{x_n}^{x_{n+1}} \cdots \int_{x_1}^{x_2}$$

$$\times \begin{vmatrix} k_2^{(1)}(z_1) \cdots k_r^{(1)}(z_1), K^{(1)}(z_1, \xi_1) \cdots K^{(1)}(z_1, \xi_{n-r}), & -\frac{1}{x_2 - x_1} \\ \cdot \qquad \cdot & +\frac{1}{x_3 - x_2} \\ \cdot \qquad \cdot & \cdot \\ \cdot \qquad \cdot & \cdot \\ k_2^{(1)}(z_n) \cdots k_r^{(1)}(z_n), K^{(1)}(z_n, \xi_1) \cdots K^{(1)}(z_n, \xi_{n-r}), & \frac{(-1)^n}{x_{n+1} - x_n} \end{vmatrix}$$

$$\times \, dz_1 \cdots dz_n,$$

where $K^{(1)}(z, \xi) = \partial K(z, \xi)/\partial z$, $k_j^{(1)}(z) = dk_j(z)/dz$.

In the text of the proof of Lemma 2 we have shown that each cofactor of a element in the last column of $\hat{J}$ is non-negative and indeed by Remark 2 for $z_i = x_i$ $(i = 2,..., n)$ each such cofactor is strictly positive. Hence $\hat{J} \neq 0$ and equivalently $J \neq 0$. Now as in the *Uniqueness of Interpolation Theorem* we can use the smoothing parameter ε in the *implicit function theorem* to establish the existence of an extended totally positive solution, $P^{(\epsilon)}(t)$, to (10) and (11) in a neighborhood of $\varepsilon = 0$, $E = E^*$, $A = A^*$ and $\{x_i = x_i^*\}_{i=2}^n$. Here

$$\begin{aligned} P^{(\epsilon)}(x_i^{(\epsilon)}) &= (-1)^{i-r+1} E^{(\epsilon)} \qquad && (i = 1,..., n + 1), \\ \frac{dP^{(\epsilon)}}{dx}(x_i^{(\epsilon)}) &= 0 && (i = 2,..., n - 1) \end{aligned} \tag{12}$$

and

$$P^{(\epsilon)}(t) = \sum_{k=1}^{r} a_i^{(\epsilon)} k_i^{(\epsilon)}(t) + \sum_{i=0}^{n-r} (-1)^i \int_{\xi_i^{(\epsilon)}}^{\xi_{i+1}^{(\epsilon)}} K^{(\epsilon)}(t, y)\, dy.$$

Now if there were another perfect spline Q which satisfied (I) and (II) we could proceed as before to find a $Q^{(\epsilon)}$ which satisfies (12). By continuity for small $|\varepsilon| > 0$, $Q^{(\epsilon)} \not\equiv P^{(\epsilon)}$. It is easy to see that $P^{(\epsilon)} - Q^{(\epsilon)}$ can be written in the form

$$(P^{(\epsilon)} - Q^{(\epsilon)})(t) = \sum_{i=1}^{r} b_i^{(\epsilon)} k_i^{(\epsilon)}(t) + \int_0^1 h^{(\epsilon)}(y)\, K^{(\epsilon)}(t, y)\, dy,$$

where $h^{(\epsilon)}(y)$ has at most $(n-r-1)$ sign changes. But from the alternation properties of $P^{(\epsilon)}$ and $Q^{(\epsilon)}$, the difference has at least n zeros. This contradicts Lemma 1. Thus $P \equiv Q$ and the proof is complete. ∎

Remark. If $k_1 > 0$ and $Dk_2 > 0$, a proof following the outline of the above theorem can be constructed to show that there is exactly one P which satisfies (I) and (II) where $\pm E$ is replaced by $\pm Ek_1(t)$ in (I) and E is replaced by $Ek_1(x_i)$ in the ith equation in (II.)

COROLLARY 3. *The perfect spline of* Theorem 5 *is the unique perfect spline (of the specified form) of minimum norm.*

Proof. Let $P(t)$ be the unique perfect spline described in Theorem 5 with $|P(x_i)| = E$, $i = 1,..., n+1$. Let $Q(t)$ be another perfect spline of the specified form with $\max_{t \in [0,1]} |Q(t)| = E$. We will show $Q(t) \equiv P(t)$.

We first note that there at most $n+1$ points where $Q(t)$ assumes the value E. This follows since $Q'(t)$ can vanish at most $n-1$ times.

If $Q(t)$ assumes the value E at exactly $n+1$ points, then by Theorem 5, $Q(t) \equiv P(t)$.

On the other hand, we now show that it is impossible for $Q(t)$ to assume the value E at fewer than $n+1$ points.

If this happened there is at least one point x_k, where $|P(x_k)| = E > |Q(x_k)|$. It then follows, as in the proof of Theorem 3 that the Jacobian

$$\frac{\partial(P(x_1),..., P(x_{k-1}), P(x_{k+1}),..., P(x_n))}{\partial(a_1,..., a_r, \xi_1,..., \xi_{n-r})}$$

$$= K\begin{pmatrix} 1,..., & r, \xi_1,..., & \xi_{n-r} \\ x_1,..., & x_{k-1}, x_{k+1},..., & x_{n+1} \end{pmatrix}$$

$$= \int_{x_n}^{x_{n+1}} \cdots \int_{x_{k-1}}^{x_{k+1}} \int_{x_{k-2}}^{x_{k-1}} \cdots \int_{x_1}^{x_2} \begin{vmatrix} k_2^{(1)}(z_1) & \cdots & K^{(1)}(z_1, \xi_{n-r}) \\ \vdots & & \vdots \\ k_2^{(1)}(z_n) & \cdots & K^{(1)}(z_n, \xi_{n-r}) \end{vmatrix}$$

$$\times\, dz_1 \cdots dz_n \neq 0.$$

Since the Jacobian does not vanish, it follows that there exists a perfect spline $P_1(t)$ of the specified form satisfying

$$|P_1(x_i)| > |Q(x_i)| \qquad (i = 1,..., n+1).$$

$$\operatorname{sgn} P_1(x_i) = -\operatorname{sgn} P_1(x_{i+1}) \qquad (i = 1,..., n).$$

Hence for sufficiently small ε the smooth function $P_{1\epsilon}(t) - Q_\epsilon(t)$ has at least n zeros. As mentioned in the last paragraph of the proof of Theorem 5, this is a contradiction. Hence the Corollary follows. ∎

Micchelli has communicated to us an elegant proof of the uniqueness theorem of Karlin [5] on oscillating L-Splines.

THE INTERPOLATING ENVELOPE

In the last several years there has been a great deal of interest in obtaining upper and lower bounds for functions for which a prescribed derivative is bounded and whose values are known over a specified point set. The basic results in this area are due to Micchelli and Miranker [13], see also [1] and [3]. These results were based on the polynomial spline kernel, $K(x, y) = (x - y)_+^{r-1}$, and its extension on L-splines [5]. These kernels have the property that they are totally positive [6, p. 11]. Micchelli and Pinkus [12, example 4] have recently given an interesting example of a totally positive kernel which is not covered by these previous results. In this section we extend these results to complete totally positive kernels; that is kernels which satisfy our *Basic Assumptions.*

THEOREM 6. *Assume in* Basic Assumption 3 *that*

$$K\begin{pmatrix} i_1,..., i_s, y_1,..., y_l \\ t_1,..., t_s, t_{s+1},..., t_{s+l} \end{pmatrix}$$

is always positive. Consider a function of the form

$$Q(x) = \sum_{i=1}^{r} a_i k_i(x) + \int_0^1 h(y) K(x, y)\, dy, \tag{13}$$

where $h \in L^\infty[0, 1]$ *and* $m < h(x) < M$ *a.e. for* $x \in [0, 1]$. *Then for the given points* $0 < x_1 < \cdots < x_n < 1$ *there exist two unique functions*

$$P_M = \sum_{i=1}^{r} b_i k_i(x) + \int_0^1 h_M(y) K(x, y)\, dy, \tag{14}$$

$$P_m(x) = \sum_{i=1}^{r} c_i k_i(x) + \int_0^1 h_m(y) K(x, y)\, dy \tag{15}$$

such that

(a) h_M *and* h_m *are step functions each with exactly* $n - r$ *jumps.*

(b) *Each of these step functions satisfies*

$$(h(x) - m)(h(x) - M) = 0 \qquad a.e.$$

(c) $h_M(x) = M$ *in some interval beginning at zero.*

(d) *$h_m(x) = m$ in some interval beginning at zero.*

(e) $P_M(x_i) = Q(x_i) = P_m(x_i)$, $i = 1,\ldots, n$.

Proof. Consider the compound kernel

$$J(x, y) = K\begin{pmatrix} 1,\ldots, r, y \\ x_1,\ldots, x_r, x \end{pmatrix} \Big/ K\begin{pmatrix} 1,\ldots, r \\ x_1,\ldots, x_r \end{pmatrix}.$$

Thus $J(x_{r+i}, y)$ is a linear combination of, or rather a linear operator on, the $K(x_j, y)$. Applying this operator to a function $y = P_M$ of the form (14) with the data, $((x_i, Q(x_i))$; we arrive at the system,

$$l_i = \int_0^1 h_M(y)\, J(x_i, y)\, dy \qquad (i = r+1,\ldots, n), \tag{16}$$

with l_i a linear combination of the $Q(x_j)$.

By Sylvester's determinant identity [6, p. 3] for $0 < y_1 < y_2 < \cdots < y_{n-r} < 1$,

$$J\begin{pmatrix} x_{r+1},\ldots, x_{n-r} \\ y_1,\ldots, y_{n-r} \end{pmatrix} = \frac{K\begin{pmatrix} 1,\ldots, r, y_1,\ldots, y_{n-r} \\ x_1,\ldots, x_r, x_{r+1},\ldots, x_n \end{pmatrix}}{K\begin{pmatrix} 1,\ldots, r \\ x_1,\ldots, x_r \end{pmatrix}}.$$

Thus $\{J(x_1, y),\ldots, J(x_{n-r}, y)\}$ form an $n-r$ dimensional Tchebycheff system. By a theorem of Krein, see [14, Theorem F], there is an h_M which yields a solution to (16) and which also satisfies (a), (b) and (c). Since $\{k_i(x)\}_{i=1}^r$ form a Tchebycheff system, a unique set of coefficients $\{b_i\}_{i=1}^r$ exist for which

$$Q(x_j) = \sum_{i=1}^r b_i k_i(x_j) + \int_0^1 h_M(y)\, K(x_j, y)\, dy \qquad (j = 1,\ldots, r).$$

Letting

$$P_M(x) = \sum_{i=1}^r b_i k_i(x) + \int_0^1 h_M(y)\, K(x, y)\, dy,$$

we see that P_M satisfies for $r+1 \leqslant i \leqslant n$,

$$\frac{K\begin{pmatrix} 1,\ldots, r, P_M \\ x_1,\ldots, x_r, x_i \end{pmatrix}}{K\begin{pmatrix} 1,\ldots, r \\ x_1,\ldots, x_r \end{pmatrix}} = \int_0^1 h_M(y)\, J(x_i, y)\, dy = \frac{K\begin{pmatrix} 1,\ldots, r, Q \\ x_1,\ldots, x_r, x_i \end{pmatrix}}{K\begin{pmatrix} 1,\ldots, r \\ x_1,\ldots, x_r \end{pmatrix}}.$$

Solving for $P_M(x_i)$ on the left and $Q(x_i)$ on the right, one finds that

$$P_M(x_i) = Q(x_i).$$

Thus P_M also satisfies (e). If there were another function P of the desired form, then $P_M - P$ could be represented as

$$P_M(x) - P(x) = \sum_{i=0}^{k} c_i k_i(x) + \int_0^1 h(x) K(x, y)\, dy,$$

where $h(x)$ has at most $(n-r-1)$ sign changes and $h(x)$ is non-zero on a set of positive measure. But $P_M - P$ has at least n zeros. This contradicts Lemma 1, case A. Thus P_M is unique. A similar procedure shows that there is a unique P_m with the desired properties. ∎

DEFINITION. A function P is called an (m, M) perfect spline with s knots $0 < y_1 < \cdots < y_s < 1$ if P can be written in the form

$$P(x) = \sum_{i=1}^{r} a_i k_i(x) + \int_0^1 h(y) K(x, y)\, dy,$$

where h is a step function with s jumps occurring at $y_1, ..., y_s$. Further if $0 = y_0$, and $1 = y_{s+1}$; $x \in [y_0, y_1] \Rightarrow h(x) = m$; $x \in [y_1, y_2] \Rightarrow h(x) = M$, etc.

Clearly any (m, M) spline $P(x)$ with a fixed number of knots, say, $n-r$ can be parametrized using the vector $A = (a_1, ..., a_r, y_1, ..., y_{n-r})$. We indicate this by letting $P(x) \equiv P(A, x)$.

Note that

$$\frac{\partial P(A, x)}{\partial y_i} = [h(y_i^-) - h(y_i^+)]\, K(x, y_i) = (-1)^{i+1}(m - M)\, K(x, y_i),$$

where of course $(-, +)$ refer to lower and upper limits, respectively. Further

$$\frac{\partial P(A, x)}{\partial a_j} = k_j(x).$$

Hence the Jacobian determinant J of $P(A, x)$ with respect to the components of A over the set of n points $0 < x_1 < \cdots < x_n < 1$, has the form

$$J = cK \begin{pmatrix} 1, ..., r, y_1, ..., y_{n-r} \\ x_1, ..., x_r, x_{r+1}, ..., x_n \end{pmatrix}, \tag{17}$$

where $c = (-1)^{1/2(n-r)(n-r+1)}(m - M)^{n-r}$.

LEMMA 4. *If $h \in L^\infty$ is non-zero a.e. and further has exactly $n-r$ sign changes occurring at $0<\xi_1<\cdots<\xi_{n-r}<1$, and if the function*

$$g(x)=\sum_{i=1}^{r} a_i k_i(x)+\int_0^1 h(y)\,K(x,y)\,dy$$

has n distinct zeros $0<x_1<\cdots<x_n<1$ then

$$K\begin{pmatrix}1,\ldots,r,\ y_1,\ldots,\ y_{n-r}\\ x_1,\ldots,x_r,\ x_{r+1},\ldots,x_n\end{pmatrix}>0.$$

Proof. The method of proof of Lemma 7.2 of [12] applied to Lemma 1, Case B carries over the the present situation.

LEMMA 5. *Consider any $0<x_1<\cdots<x_n<1$; $m<M$ and any Q of the form (13) with the corresponding h satisfying $m<h<M$. Then any perfect (m,M) or (M,m) spline P with s knots $0<y_1<\cdots<y_s<1$, $s\leqslant n-r$, which interpolates Q over the point set $\{x_i\}_{i=1}^n$ has the properties that $s=n-r$ and*

$$K\begin{pmatrix}1,\ldots,r,\ y_1,\ldots,\ y_{n-r}\\ x_1,\ldots,x_r,\ x_{r+1},\ldots,x_n\end{pmatrix}>0. \tag{18}$$

Proof. If h_p and h_Q are the "h functions" corresponding the P and Q, respectively, it is easy to see that h_P-h_Q has at most s sign changes and does not vanish on a set of positive measure. But $P-Q$ has at least n zeros; hence, by Lemma 1, Case B, h_p-h_Q has $n-r$ sign changes, i.e., $s=n-r$. The result now follows from Lemma 4. ∎

THEOREM 7. *Under the hypothesis of Lemma 5, there exists at most one (M,m) perfect spline P_M, with the number of knots not exceeding $n-r$, which interpolates Q over the point set $\{x_i\}_{i=1}^n$. Indeed P_M has exactly $n-r$ knots. A similar result holds for an (m,M) perfect spline P_m.*

Proof. Assume that there is an (M,m) perfect spline P_M with the desired properties. By Lemma 1, Case B, P_M has $n-r$ knots. Let $A^*=(a_1^*,\ldots,a_r^*,\ldots,a_r^*,\ y_1^*,\ldots,\ y_{n-r}^*)$ be the parameter vector associated with P_M. By Lemma 5,

$$K\begin{pmatrix}1,\ldots,r,\ y_1^*,\ldots,\ y_{n-r}^*\\ x_1,\ldots,x_r,\ x_{r+1},\ldots,x_n\end{pmatrix}>0. \tag{19}$$

Assume there exists another (M,m) perfect spline $P(x)$ with at most $n-r$ knots which interpolates Q over the point set. By Lemma 1,, Case B, P has exactly $n-r$ knots.

Proceeding as in the proof of Theorem 2, we find by using the *implicit* function theorem that the perturbed Gaussian Transform of $P_M(x)$ can be made to agree with the Gaussian Transform of $P(x)$ at x_i, $i = 1,\dots, n$. By Lemma 1 this shows $P_M(x) = P(x)$. The result for $P_m(x)$ is proved similarly. ∎

LEMMA 6. *Assume in* Basic Assumption 3 *that*

$$K\begin{pmatrix} i_1,\dots, i_s, y_1,\dots, y_l \\ t_1,\dots, t_s, t_{s+1},\dots, t_{s+l} \end{pmatrix}$$

is always positive and that $h(y)$ *is a non-zero step function with at most* $n-r$ *sign changes. Then any function of the form*

$$P(x) = \sum_{i=1}^{r} a_i k_i(x) + \int_0^1 h(y)\, K(x, y)\, dy$$

has at most n *zeros including multiplicities in* $(0, 1)$.

Proof. See Proposition 3.1 [16].

THEOREM 8. *Consider* $0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1$; $Q_0 \in R^n$ *and* $m < M$. *Then there exists a unique* (M, m) *perfect spline* P_M *and a unique* (m, M) *perfect spline* P_m *each with at most* $n-r$ *knots such that for any* Q *of the form* (13) *with the corresponding* h *satisfying*

$$m < h < M \qquad \text{a.e.}$$

and $\bar{Q}(x) \equiv (Q(x_1),\dots, Q(x_n)) = Q_0$, *the following propositions are valid*:

$$(-1)^{r+i}(P_M(x) - Q(x)) > 0, \qquad x_i < x < x_{i+1}\ (i = 0, 1,\dots, n), \tag{20}$$

$$(-1)^{r+i}(P_m(x) - Q(x)) < 0, \qquad x_i < x < x_{i+1}\ (i = 0, 2,\dots, n), \tag{21}$$

$$P_m \text{ and } P_M \text{ have exactly } n-r \text{ knots.} \tag{22}$$

Proof. We employ a perturbation technique involving the Gaussian Transform. Consider a strictly decreasing sequence $\{\varepsilon_\nu\}$ converging to zero. For each ν, let $P_{M\nu}(x)$ be the unique (M, m) perfect spline guaranteed by Theorem 7 of the form

$$P_{M\nu}(x) = \sum_{i=1}^{r} a_{i\nu}(k(x; \varepsilon_\nu)) + \int_0^1 h_\nu(y)\, K(x, y; \varepsilon_\nu)\, dy$$

which has exactly $n-r$ knots and for which $\overline{P_{M\nu}(x)} = \overline{Q(x; \varepsilon_\nu)}$. If h_Q is the "h function" associated with $Q(x)$ (and this with $Q(x; \varepsilon_\nu)$), it is easy to see that $h_\nu - h_Q$ is non-zero a.e. and has exactly $n-r$ sign changes. Using a perturbation technique, one can show $P_{M\nu}(x) - Q(x;\varepsilon)$ has n simple zeros in $(0, 1)$. Hence the zeros of $P_{M\nu}(x) - Q(x;\varepsilon_\nu)$ are simple occurring at $0 < x_1 < \cdots < x_n < 1$. Thus by the generalized sign properties of perfect splines [6, pp. 232–233], $P_{M\nu}(x) - Q(x;\varepsilon_\nu)$ must satisfy (20).

We claim that the set $\{a_{i\nu}\}_{i=1}^{n}{}_{\nu=1}^{\infty}$ is bounded. If not we can arrive at the situation that for some set of $\{b_i\}_{i=1}^r$, where $\sum_{i=1}^r |b_i| > 0$

$$\sum_{i=1}^{r} b_i k_i(x_j) = 0 \qquad (j = 1,..., n),$$

which contradicts that assumption that $k_1,..., k_r$ form a Tchebycheff system. Thus the set of coefficients is bounded. Hence by going to a subsequence (which we do not relabel), we can find sequences $\{P_{M\nu}\}$ and $\{Q(\cdot, \varepsilon_\nu)\}$ such that there is a perfect spline $P(x)$ with the properties:

(1) The $h(x)$ associated with $P(x)$ is a step function with at most $n-r$ jumps satisfying $(h(x) - M)(h(x) - m) = 0$ a.e.

(2) $\lim_{\nu\to\infty} \|P - P_{M\nu}\| = 0$, where $\|\cdot\|$ is the uniform norm over $[0, 1]$.

(3) $\overline{P(x)} = \lim_{\nu\to\infty} \overline{Q(x, \varepsilon_\nu)} = \bar{Q}$.

(4) By Lemma 1, Case B, $P_M - Q = 0$ only at the interpolating points and thus P satisfies (20).

By Theorem 7, h has exactly $n-r$ jumps, and $P \equiv P_M$. In a similar fashion we can show P_M also satisfies (21) and (22). ∎

COROLLARY 4. *If we weaken the hypotheses of Theorem* 3 *to allow* $m \leqslant h \leqslant M$, *the conclusions of Theorem* 8 *are still valid except that* P_m *and* P_M *have at most* $n-r$ *knots.*

Proof. Consider a strictly decreasing sequence $\{M_\nu\}$ converging to M and a strictly increasing sequence $\{m_\nu\}$ converging to m. By Theorem 8 there is an (M_ν, m_ν) perfect spline P_{M_ν} which satisfies (20) and (22). Using an argument similar to the one applied in Theorem 8, we can find a limit (M, m) perfect spline P_M which satisfies (20), (22) and which has at most $n-r$ knots. Clearly the same technique can be used to find P_m. ∎

For polynomial perfect splines, Theorem 8 and its corollary were first proven by Micchelli and Miranker [13] and elaborated upon by Micchelli and Rivlin [14]. Another proof was given by de Boor [1]. The result was recently rediscovered by Lee and Goodman [3]. In these papers some of the x_i's were permitted to coincide. Our results can be extended to permit this by letting the $\{k_i\}_{i=1}^r$ satisfy additional conditions. For example, if $k_1 > 0$, we can allow at most two x_i's to agree; if in addition

$$\frac{d}{dx}\left(\frac{k_2(x)}{k_1(x)}\right) > 0$$

we can permit three x_i's to agree and so forth.

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